Counting the Scaled +1/-1 Matrices that Satisfy the Restricted Isometry Property

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Abstract

An $m \times n$ real matrix Φ is said to satisfy the Restricted Isometry Property (RIP) of order k if it nearly preserves the ℓ_2-norm of all vectors in $\mathbb{R}^n$ that have no more than k nonzero entries. Matrices that satisfy the RIP are useful in compressed sensing, a fast-expanding field of mathematics and signal processing. The $m \times n$ matrices with entries $+1$ and -1 and scaled by $1/\sqrt{m}$ are often mentioned as an example of matrices that satisfy the RIP with high probability. We show that if $n \leqslant 2^{m-1}$, then the precise count of these matrices that satisfy the RIP of order $k = 2$ is

$$\rho(m, n) \quad = \quad 2^n \prod_{j=0}^{n-1} \left(2^{m-1} - j\right)$$

and the restricted isometry constants do not exceed $1 - 2/m$. If $n > 2^{m-1}$, then there are no scaled +1/-1 matrices of size $m \times n$ that satisfy the RIP, except for the order $k = 1$.

Key words. Restricted Isometry Property; Compressed Sensing.

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1 Introduction

Compressive sensing consists in recovering large but sparse encoded datasets from small datasets. This is known to be feasible when the encoding is linear through a matrix that satisfies the Restricted Isometry Property. Compressive sensing is a rapidly expanding field in mathematics, statistics, computer science and signal processing. There is abondant and fast-growing literature.

The survey article of Candès [2] provides a good coverage of the topic and references. The Digital Signal Processing group at Rice University maintains a comprehensive online portal of compressive sensing resources [6].

A real matrix is said to satisfy the Restricted Isometry Property if it nearly preserves the ℓ_2-norm of sparse vectors. For specificity, let Φ be an $m \times n$ real matrix, let $\delta \in \mathbb{R}_{\geqslant 0}$, and let $k \in [1..n]$. The matrix Φ satisfies the Restricted Isometry Property of order k with constant δ, or more concisely, is $\mathrm{RIP}(k, \delta)$, if for every column vector $x \in \mathbb{R}^{n,1}$, we have

$$(1-\delta)\,\|x\|^2_{\ell_2^n} \;\leqslant\; \|\Phi x\|^2_{\ell_2^m} \;\leqslant\; (1+\delta)\,\|x\|^2_{\ell_2^n}\ , \tag{1.1}$$

provided the vector x is k-sparse, i.e. has no more than k nonzero entries. For any subset $T \subseteq [1..n]$, let Φ_T denote the submatrix of Φ supported on $[1..m] \times T$. Then the matrix Φ is $\mathrm{RIP}(k, \delta)$ if and only if for every subset $T \subseteq [1..n]$ of cardinality $|T| = k$, the eigenvalues of the matrix $\Phi_T^{\mathrm{t}}\Phi_T$, a $k \times k$ symmetric positive-semidefinite matrix, lie in the interval $[1-\delta, 1+\delta]$. The Restricted Isometry Constant of order k of Φ, denoted $\delta_k(\Phi)$, is the smallest $\delta \in \mathbb{R}_{\geqslant 0}$ such that Φ is $\mathrm{RIP}(k, \delta)$. Evidently, with σ denoting the spectrum of a square matrix, we have

$$\delta_k(\Phi) \;=\; \max\left\{ \left|1-\lambda\right| : \lambda \in \sigma\left(\Phi_T^{\mathrm{t}}\Phi_T\right), T \subseteq [1..n], |T| = k \right\}. \tag{1.2}$$

The matrix Φ satisfies the Restricted Isometry Property of order k, or more concisely is $\mathrm{RIP}(k)$, if $\delta_k(\Phi) < 1$, i.e. it if is $\mathrm{RIP}(k, \delta)$ for some $\delta < 1$. The importance of the RIP lies in that a RIP matrix enables compressed sampling. Candès and Tao [3] have provided RIP-based conditions that ensure that a sparse vector x is deterministically recoverable as the unique solution of minimal ℓ_1^n-norm of the equation $\Phi x = y$.

Equation (1.2) shows that determining whether a particular matrix is $\mathrm{RIP}(k)$ will generally be a computing-intense effort. The most common examples of RIP matrices in the literature are asserted to be so with high probability, but not determiniscally. See Baraniuk, Davenport, DeVore and Wakin [1] for the three classical examples. The ability to deterministically establish whether matrices of interest are RIP is the subject of current concern and ongoing efforts. At the time of this writing, the respected blog of Terence Tao mentions this as an open problem [5]. Developments in this area include the work of DeVore [4].

Because

$$\delta_1(\Phi) \leqslant \delta_2(\Phi) \leqslant \cdots \leqslant \delta_{n-1}(\Phi) \leqslant \delta_n(\Phi) , \tag{1.3}$$

it is helpful to know the values of $\delta_k(\Phi)$ for low values of k. For instance, if $\delta_2(\Phi) \geqslant 1$, then there is no hope that Φ would be RIP(k) for higher but more interesting orders k.

This work focuses on one of the three classical examples: the $m\times n$ matrices whose entries are $+1/\sqrt{m}$ and $-1/\sqrt{m}$. It is actually easy to calculate their restricted isometry constants of order two. Consider

$$\rho(m,n) \quad = \quad \begin{cases} 2^n \displaystyle\prod_{j=0}^{n-1} \left(2^{m-1} - j\right) & \text{if } \ n \leqslant 2^{m-1} \\ 0 & \text{if } \ n > 2^{m-1} . \end{cases}$$

We show that, of the 2^{mn} possible matrices, $\rho(m,n)$ matrices Φ have $\delta_2(\Phi) \leqslant 1 - 2/m$ and the other matrices Φ have $\delta_2(\Phi) = 1$. So if $n > 2^{m-1}$, then none of the matrices are RIP(k) with $k \geqslant 2$. (They all are RIP(1) with $\delta_1(\Phi) = 0$.)

In the rest of the paper, all inner products and norms are with respect to the applicable canonical Euclidean structure; we will no longer use indices such as ℓ_2^n.

2 The Restricted Isometry Property of Order Two

Let Φ be a real $m\times n$ matrix with column vectors $u_1, \ldots, u_n \in \mathbb{R}^{m,1}$. From equation (1.2), we readily have

$$\delta_1(\Phi) \quad = \quad \max_{1\leqslant j\leqslant n} \left| 1 - \|u_j\|^2 \right| . \tag{2.1}$$

Obtaining $\delta_2(\Phi)$ is not much more difficult if all column vectors have norm one.

Theorem 2.1. *Suppose $n \geqslant 2$ and $\|u_1\| = \cdots = \|u_n\| = 1$. Then*

$$\delta_2(\Phi) \quad = \quad \max_{1\leqslant p<q\leqslant n} \left| \langle u_p, u_q \rangle \right| . \tag{2.2}$$

Proof. Let $T \subseteq [1..n]$ with $|T| = 2$; let $p, q \in [1..n]$ such that $p < q$ and $T = \{p, q\}$. We have

$$\Phi_T^{\mathrm{t}}\, \Phi_T \;=\; \begin{pmatrix} u_p^{\mathrm{t}} \\ u_q^{\mathrm{t}} \end{pmatrix} \begin{pmatrix} u_p \ u_q \end{pmatrix} \;=\; \begin{pmatrix} \|u_p\|^2 & \langle u_p, u_q \rangle \\ \langle u_p, u_q \rangle & \|u_q\|^2 \end{pmatrix} \;=\; \begin{pmatrix} 1 & \langle u_p, u_q \rangle \\ \langle u_p, u_q \rangle & 1 \end{pmatrix} .$$

The eigenvalues of this matrix are $1 - \langle u_p, u_q \rangle$ and $1 + \langle u_p, u_q \rangle$. Hence, equation (2.2) is an instance of equation (1.2). □

Corollary 2.2. *Suppose $n \geqslant 2$ and $\|u_1\| = \cdots = \|u_n\| = 1$. If for every $p, q \in [1..n]$ with $p \neq q$, it holds that $u_p \neq u_q$ and $u_p \neq -u_q$, then $\delta_2(\Phi) < 1$ and Φ is* RIP(2). *If there exist $p, q \in [1..n]$ with $p \neq q$ such that $u_p = u_q$ or $u_p = -u_q$, then $\delta_2(\Phi) = 1$ and Φ is not* RIP(k) *for any $k \geqslant 2$.*

Proof. Given $u, v \in \mathbb{R}^{m,1}$ such that $\|u\| = \|v\| = 1$, we have $\left|\langle u, v \rangle\right| \leqslant 1$, and $\langle u, v \rangle = 1 \Leftrightarrow u = v$, and $\langle u, v \rangle = -1 \Leftrightarrow u = -v$. This remark along with Theorem 2.1 prove Corollary 2.2. □

If the columns of Φ are taken from a prescribed finite set, then the presence of duplicate or opposite columns becomes likely, and even necessary, as the number n of columns grows. Corollary 2.2 shows that the RIP of order two then becomes unlikely to impossible. It is this observation that we make explicit for the scaled +1/-1 matrices in the next section.

3 The RIP of Order Two for the Scaled +1/-1 Matrices

Let E_m be the set of column vectors in $\mathbb{R}^{m,1}$ whose entries all equal $\dfrac{+1}{\sqrt{m}}$ or $\dfrac{-1}{\sqrt{m}}$. The set E_m has 2^m elements and they all have norm one. We consider the RIP of order two for the matrices whose column vectors are in E_m.

Lemma 3.1. *Suppose $m \geqslant 2$ and let $u, v \in E_m$ such that $u \neq v$ and $u \neq -v$. Then $\left|\langle u, v \rangle\right| \leqslant 1 - \dfrac{2}{m}$.*

Proof. Because $u \neq v$, there exists $r \in [1..m]$ such that $u_r \neq v_r$, or equivalently $u_r = -v_r$; and because $u \neq -v$, there exists $s \in [1..m]$ such that $u_s \neq -v_s$, or equivalently $u_s = v_s$. The indices r and s are distinct because all entries of u and v are nonzero. Therefore we have

$$u_r\, v_r + u_s\, v_s \;=\; -u_r^2 + u_s^2 \;=\; -\frac{1}{m} + \frac{1}{m} \;=\; 0$$

and

$$\left|\langle u, v \rangle\right| \;=\; \left| \sum_{\substack{1 \leqslant i \leqslant m \\ i \neq r,s}} u_i v_i \right| \;\leqslant\; \sum_{\substack{1 \leqslant i \leqslant m \\ i \neq r,s}} \left|u_i\, v_i\right| \;=\; (m-2)\,\frac{1}{m} \;=\; 1 - \frac{2}{m}\,.$$

□

The next result is a specialized version of Corollary 2.2. It readily obtains by combining Theorem 2.1 and Lemma 3.1.

Theorem 3.2. *Suppose $m, n \geqslant 2$ and let Φ be an $m \times n$ matrix with column vectors $u_1, \ldots, u_n \in E_m$. If for every $p, q \in [1..n]$ with $p \neq q$, it holds that $u_p \neq u_q$ and $u_p \neq -u_q$, then $\delta_2(\Phi) \leqslant 1 - \dfrac{2}{m}$ and Φ is* RIP(2)*. If there exist $p, q \in [1..n]$ with $p \neq q$ such that $u_p = u_q$ or $u_p = -u_q$, then $\delta_2(\Phi) = 1$ and Φ is not* RIP(k) *for any $k \geqslant 2$.* □

Theorem 3.2 characterizes the scaled +1/-1 matrices that are RIP(2) in a fashion that allows as to count them.

Theorem 3.3. *Suppose $m, n \geqslant 2$. Let $\rho(m, n)$ be defined by*

$$\rho(m,n) \quad = \quad \begin{cases} 2^n \displaystyle\prod_{j=0}^{n-1} \left(2^{m-1} - j\right) & \text{if } n \leqslant 2^{m-1} \\ 0 & \text{if } n > 2^{m-1} . \end{cases} \tag{3.1}$$

Among the 2^{mn} matrices of size $m \times n$ whose entries are $\dfrac{+1}{\sqrt{m}}$ and $\dfrac{-1}{\sqrt{m}}$, there are $\rho(m, n)$ matrices Φ such that $\delta_2(\Phi) \leqslant 1 - \dfrac{2}{m}$; they are RIP(2)*. For the remaining matrices Φ, we have $\delta_2(\Phi) = 1$; they are not* RIP(k) *for any $k \geqslant 2$.*

Proof. For $u_1, \ldots, u_n \in E_m$, consider the the following property.

$$\mathscr{P}\big(u_1, \ldots, u_n\big) \ : \ \forall\, q \in [2..n]\,,\ \forall\, p \in [1..(q-1)]\,,\ u_q \neq u_p \text{ and } u_q \neq -u_p\,.$$

For a single vector $u \in E_m$, let $\mathscr{P}(u)$ simply be a tautology. Then:

$$\begin{aligned} &\big(\text{The matrix } \Phi = \big(u_1, \ldots, u_n\big) \text{ is RIP(2)}\big) \\ &\quad \Leftrightarrow \mathscr{P}\big(u_1, \ldots, u_n\big) \\ &\quad \Leftrightarrow \mathscr{P}\big(u_1, \ldots, u_{n-1}\big) \text{ and } \big(\forall\, j \in [1..(n-1)]\,,\ u_n \neq u_j \text{ and } u_n \neq -u_j\big) \\ &\quad \Leftrightarrow \mathscr{P}\big(u_1, \ldots, u_{n-1}\big) \text{ and } \big(u_n \in E_m \backslash \{u_1, \ldots, u_{n-1}, -u_1, \ldots, -u_{n-1}\}\big)\,. \end{aligned}$$

The instance of this equivalence for $n = 2$ yields

$$\rho(m,2) \quad = \quad 2^m \cdot \big(2^m - 2\big)\,,$$

while for $3 \leqslant n \leqslant 2^{m-1}$, it leads to

$$\rho(m,n) \quad = \quad \rho(m,n-1) \cdot \big(2^m - 2\,(n-1)\big)\,.$$

It follows that if $2 \leqslant n \leqslant 2^{m-1}$, then

$$\begin{aligned}
\rho(m,n) &= \rho(m,2) \prod_{j=2}^{n-1} \left(2^m - 2\,j\right) \\
&= 2^m \left(2^m - 2\right) \prod_{j=2}^{n-1} \left(2^m - 2\,j\right) \\
&= \prod_{j=0}^{n-1} \left(2^m - 2\,j\right) \\
&= 2^n \prod_{j=0}^{n-1} \left(2^{m-1} - j\right) .
\end{aligned}$$

If $n = 2^{m-1} + 1$ and property $\mathscr{P}\big(u_1, \ldots, u_{n-1}\big)$ is true, then the set $E_m \backslash \{u_1, \ldots, u_{n-1}, -u_1, \ldots, -u_{n-1}\}$ is empty and property $\mathscr{P}\big(u_1, \ldots, u_n\big)$ thus is false. Therefore, whenever $n > 2^{m-1}$, property $\mathscr{P}\big(u_1, \ldots, u_n\big)$ is always false, and consequently $\rho(m,n) = 0$. □

We see in equation (3.1) that the formula for $\rho(m,n)$ that applies if $n \leqslant 2^{m-1}$ actually applies for all $n \geqslant 2$.

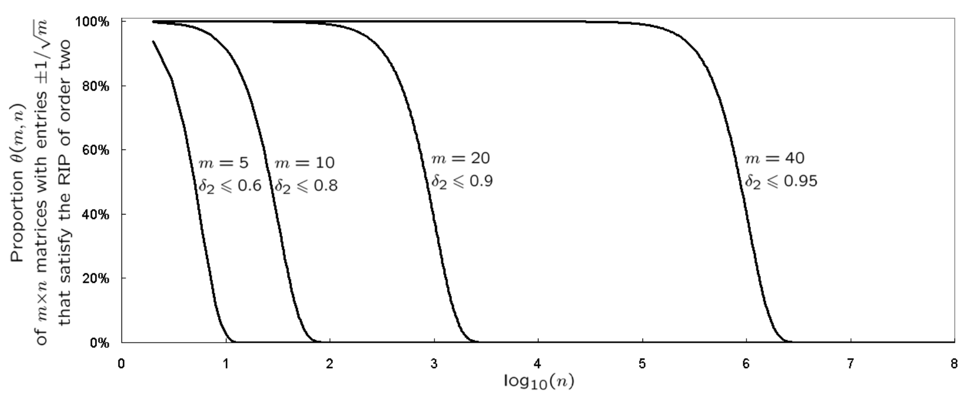

Figure 3.1: Evolution of the proportion $\theta(m,n)$ of scaled +1/-1 matrices of size $m\times n$ that satisfy the RIP of order two, with respect to the number n of columns, for four chosen values of the number m of rows. Each curve is labeled with the fixed value of m and the corresponding maximum value $1 - 2/m$ of the restricted isometry constants of order two. The horizontal axis shows the decimal logarithm values of n. The left-most points of the curves are at $n = 2$, i.e. $\log_{10}(n) = \log_{10}(2) \approx 0.3$.

We introduce the following number:

$$\theta(m,n) = \frac{\rho(m,n)}{2^{mn}} . \tag{3.2}$$

It represents the proportion of scaled +1/-1 matrices of size $m \times n$ that are RIP(2). We have:

$$\theta(m,n) = \prod_{j=1}^{n-1} \left(1 - \frac{j}{2^{m-1}}\right) . \tag{3.3}$$

The proportion $\theta(m,n)$ increases with the number m of rows and decrease with the number n of columns. Figure 3.1 helps visualize the dependence of this proportion on matrix dimensions.

4 Closing Notes

This work contributes to making the occurrence of the Restricted Isometry Property somewhat more deterministic and quantitative. We exploited properties of the set E_m: it is nonempty and finite, all its elements have norm one, and it is closed under taking opposites. The method we used readily applies whenever these conditions are in effect. We actually have the following more general result.

Theorem 4.1. *Let $m, n \in \mathbb{Z}_{\geqslant 2}$ and let E be a subset of $\mathbb{R}^{m,1}$. Suppose that E is nonempty and finite, that all elements of E have norm one, and that the opposite of each element of E is in E. Then the cardinality $|E|$ of E is even; let $M \in \mathbb{Z}_{\geqslant 1}$ such that $|E| = 2M$. Let*

$$\hat{\delta}_2(E) = \max\left\{\left|\langle u,v\rangle\right| : u,v \in E, u \neq v, u \neq -v\right\}$$

and

$$\Theta(M,n) = \prod_{j=1}^{n-1} \left(1 - \frac{j}{M}\right) .$$

Then $\Theta(M,n)$ is the proportion of the $m \times n$ matrices with column vectors from E that are RIP(2). *For these* RIP(2) *matrices Φ, we have*

$$\delta_2(\Phi) \leqslant \hat{\delta}_2(E) < 1 .$$

For all other matrices Φ, we have $\delta_2(\Phi) = 1$; they are not RIP(k) *for any order $k \geqslant 2$. If $n > M$, then no $m \times n$ matrices with column vectors from E are* RIP(k) *for any order $k \geqslant 2$.* □

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