Refilling Fixed-Capacity Containers

Gilles Gnacadja
Research and Development Information Systems, Amgen
Thousand Oaks, California, USA
http://math.GillesGnacadja.info/

California State University Channel Islands

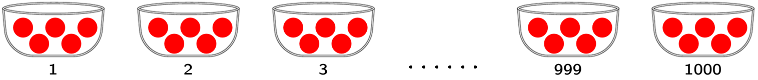
Summer Mathematics
Research Experience for Undergraduates

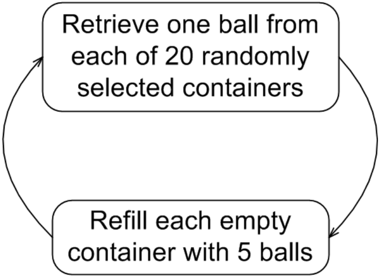
18 July 2012

Outline

Outline

Retrieving and Refilling





How many containers are refilled at the nth pass?

$n = 1$

$n = 2$

$n = 3$

$n = 4$

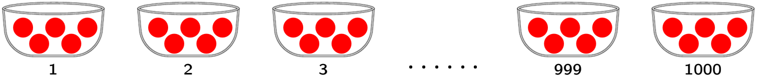
$n = 5$

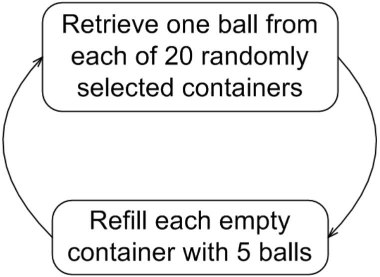
$n = 6$

$\vdots$

$n \to \infty$

Retrieving and Refilling

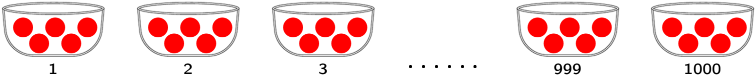


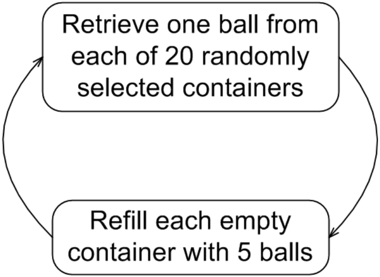


How many containers are refilled at the nth pass?

$n = 1$ refilled $= 0$
$n = 2$ refilled $= 0$
$n = 3$ refilled $= 0$
$n = 4$ refilled $= 0$
$n = 5$
$n = 6$
$\vdots$
$n \to \infty$

Retrieving and Refilling

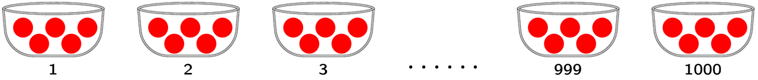


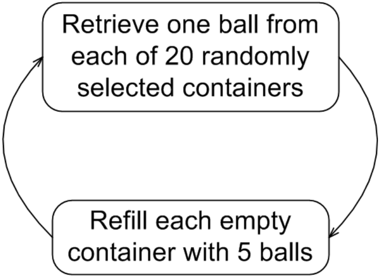


How many containers are refilled at the nth pass?

$n = 1$ refilled $= 0$
$n = 2$ refilled $= 0$
$n = 3$ refilled $= 0$
$n = 4$ refilled $= 0$
$n = 5$ refilled $= ?$
$n = 6$ refilled $= ?$
$\vdots$
$n \to \infty$

Retrieving and Refilling





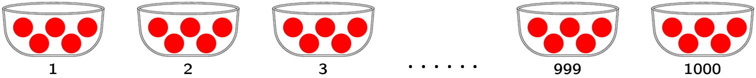
How many containers are refilled at the nth pass?

$n = 1$ refilled $= 0$
$n = 2$ refilled $= 0$
$n = 3$ refilled $= 0$
$n = 4$ refilled $= 0$
$n = 5$ refilled $= ?$
$n = 6$ refilled $= ?$
$\vdots$
$n \to \infty$ refilled $\to 4$

Goal: Prove this

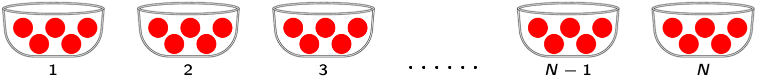
Outline

Refilling Events



Container i is refilled at the nth pass	$=$	The cumulative number of times container i has been selected since the first pass <u>just became</u> a multiple of $d = 5$ at the nth pass.
	$\subsetneq$	The cumulative number of times container i has been selected since the first pass <u>is</u> a multiple of $d = 5$ at the nth pass.

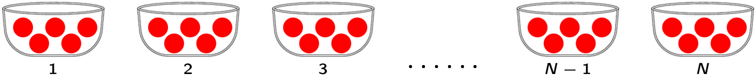
Refilling Events



Container i is refilled at the nth pass	$=$	The cumulative number of times container i has been selected since the first pass just became a multiple of d at the nth pass.

$=$ Container i is selected at the nth pass $\times$ The cumulative number of times container i has been selected at passes $1, \ldots, n-1$ is congruent to $d-1$ modulo d.

Probability of Refilling Events



Container i is refilled at the nth pass	=	Container i is selected at the nth pass	×	The cumulative number of times container i has been selected at passes $1, \ldots, n-1$ is congruent to $d-1$ modulo d.

probability $= \dfrac{M}{N}$

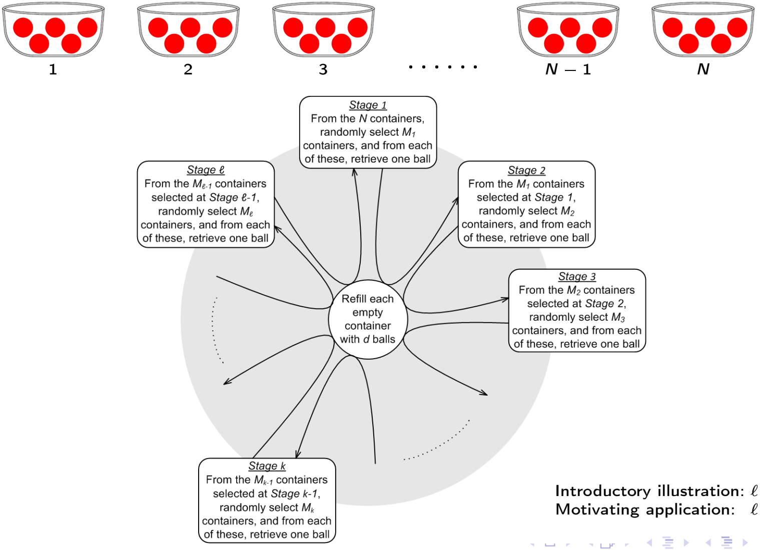
($M := \#(\text{Selected})$)

probability $\underset{n \to \infty}{\longrightarrow} \dfrac{1}{d}$

(from upcoming theorem)

Outline

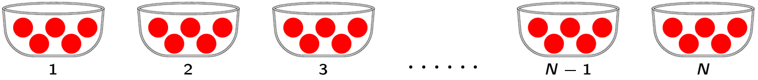
Multi-Stage Retrieving and Refilling



Introductory illustration: $\ell = 1$
Motivating application: $\ell = 2$

Outline

Probability of Refilling Events



Container i is refilled after stage k of the nth pass	=	Container i is selected at stages $1, \ldots, k$ of the nth pass	×	The cumulative number of times container i has been selected at passes $1, \ldots, n-1$ is congruent to $d-k$ modulo d.

probability $= \cdots$

probability $\xrightarrow[n\to\infty]{} 1/d$

(from upcoming theorem)

Outline

Probability Vector

$f(k)$ for $k = 0, \ldots, \ell$: Probability that, in one pass, the number of times container i is selected is k

$f = \big(f(0), \ldots, f(\ell)\big)$: Probability vector, i.e. $f(0), \ldots, f(\ell) \in \mathbb{R}_{\geqslant 0}$ and $f(0) + \cdots + f(\ell) = 1$

Exercise: $f(k) = (M_k - M_{k+1})/N$ for $k = 0, \ldots, \ell$, with $M_0 := N$ and $M_{\ell+1} := 0$

Convolution of Probability Vectors

Probability that, in n passes,
the number of times container i is selected is k

n	Range of k	Probability
1	$0 \leqslant k \leqslant \ell$	$f(k)$
2	$0 \leqslant k \leqslant 2\ell$	$(f * f)(k) = f(0)f(k) + f(1)f(k-1) + f(2)f(k-2) + \cdots$ $\cdots + f(k-1)f(1) + f(k)f(0)$
$\vdots$		
n	$0 \leqslant k \leqslant n\ell$	$(\underbrace{f * f * \cdots * f}_{n})(k) = f^{*n}(k)$

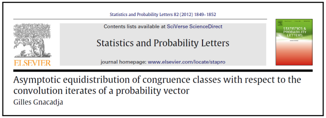
Convolution of vectors is like
multiplication of univariate polynomials.

Probability of Congruence Classes of Refilling Counts

Probability $\varphi(f, n, d, r)$ that, in n passes, the number of times container i is selected is congruent to r modulo d

$$\varphi(f, n, d, r) = \sum_{\substack{0 \leqslant k \leqslant n\ell \\ k \equiv r \bmod d}} f^{*n}(k) = \sum_{q=0}^{\mathrm{floor}((n\ell - r)/d)} f^{*n}(r + dq)$$

Asymptotic Equidistribution of Congruence Classes


Statistics and Probability Letters 82 (2012) 1849–1852
Contents lists available at SciVerse ScienceDirect
Statistics and Probability Letters
journal homepage: www.elsevier.com/locate/stapro
Asymptotic equidistribution of congruence classes with respect to the convolution iterates of a probability vector
Gilles Gnacadja

http://dx.doi.org/10.1016/j.spl.2012.05.025

Theorem

We have $\lim_{n\to\infty} \varphi(f, n, d, r) = 1/d$, where

- $f = \big(f(0), \ldots, f(\ell)\big)$ is a positive probability vector with $\ell \geqslant 1$;
- d and r are integers with $d \geqslant 1$ and $0 \leqslant r \leqslant d - 1$.

The message: ***Congruence is asymptotically equidistributed regardless of (positive) starting distribution.***

Remark: Result may, but need not, fail if starting distribution f is nonnegative but not positive.

Outline

Setting up for the Proof

Theorem to prove says that $\lim_{n\to\infty} \varphi(f,n,d)$ is the d vector filled with $1/d$, where

$$\varphi(f,n,d) := \big(\varphi(f,n,d,0), \varphi(f,n,d,1), \varphi(f,n,d,2), \ldots, \varphi(f,n,d,d-1)\big) .$$

$$\Phi(f,m,d) := \begin{pmatrix} \varphi(f,m,d,0) & \varphi(f,m,d,1) & \varphi(f,m,d,2) & \cdots & \varphi(f,m,d,d-1) \\ \varphi(f,m,d,d-1) & \varphi(f,m,d,0) & \varphi(f,m,d,1) & \cdots & \varphi(f,m,d,d-2) \\ \varphi(f,m,d,d-2) & \varphi(f,m,d,d-1) & \varphi(f,m,d,0) & \cdots & \varphi(f,m,d,d-3) \\ \vdots & \vdots & \vdots & & \vdots \\ \varphi(f,m,d,1) & \varphi(f,m,d,2) & \varphi(f,m,d,3) & \cdots & \varphi(f,m,d,0) \end{pmatrix}$$

$$= \mathrm{CirculantMatrix}\big(\varphi(f,m,d)\big)$$

- $\varphi(f,m,d)$ is a nonnegative probability vector; therefore...
- $\Phi(f,m,d)$ is a nonnegative doubly stochastic matrix.
- For m large enough (specifically $m \geqslant (d-1)/\ell$):
  - The probability vector $\varphi(f,m,d)$ is positive; therefore...
  - The doubly stochastic matrix $\Phi(f,m,d)$ is positive; therefore...
  - Perron-Frobenius Theory:
    Powers of $\Phi(f,m,d)$ converge of the $d \times d$ matrix filled with $1/d$.

Elements of Proof

Proposition For $d, m, n \in \mathbb{Z}_{\geqslant 1}$ with $n > m$,
$$\varphi(f, n, d) = \varphi(f, n-m, d) \cdot \Phi(f, m, d).$$

Corollary For $d, m, s, t \in \mathbb{Z}_{\geqslant 1}$,
$$\varphi(f, sm+t, d) = \varphi(f, t, d) \cdot \big(\Phi(f, m, d)\big)^s.$$

Consequence For $d, m, t \in \mathbb{Z}_{\geqslant 1}$ with $m \geqslant (d-1)/\ell$,
$$\varphi(f, sm+t, d) \xrightarrow[s\to\infty]{} \underbrace{(1/d, \ldots, 1/d)}_{d}.$$

Conclusion For $d \in \mathbb{Z}_{\geqslant 1}$, $\varphi(f, n, d) \xrightarrow[n\to\infty]{} \underbrace{(1/d, \ldots, 1/d)}_{d}$.