Map Transformation to Force Convergence to Unique Fixed Point

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1 The Question

Is there a transformation $\mathcal{T}$ of maps $\mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$ with the following property?

> If a map $F : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$ is smooth and order-reversing and possesses a unique fixed point ω, then
>
> - The point ω is the unique fixed point of $\mathcal{T}F$; and
> - For some (known) point $a \in \mathbb{R}^n_{\geqslant 0}$, the sequence of iterates of $\mathcal{T}F$ starting at a converges to the (unknown) point ω.

2 Why this Question?

Consider a map $F : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$ which is smooth and order-reversing and possesses a unique fixed point ω. The sequence $\big(x(k)\big)_{k\geqslant 0}$ of iterates of F with $x(0) = 0 \in \mathbb{R}^n$ satisfies the following property.

$$x(0) \leqslant x(2) \leqslant x(4) \leqslant \cdots \leqslant \omega \leqslant \cdots \leqslant x(5) \leqslant x(3) \leqslant x(1).$$

The sequence $\big(x(k)\big)_{k\geqslant 0}$ either converges to ω or accumulates to a 2-orbit $\{\omega^-, \omega^+\}$ of F such that $\omega^- \lneqq \omega \lneqq \omega^+$. ($\omega^-$ and ω^+ are sometimes referred to as coupled fixed points, but note that they are fixed points of F^2, not of F.)

I know several fixed-point-preserving transformations. However, as far as I know, these may accelerate already assured convergence; they do not guarantee convergence that is not already assured.

3 The Trivial Case $n = 1$

If $n = 1$, then the assumption that F has a unique fixed point is redundant and the problem is solved with $\mathcal{T}F$ a suitable convex combination of F and the identity map.

4 Where Does This Question Come From?

I have a smooth bijection $f = (f_1, \ldots, f_n) : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$ such that

$$\forall\, i = 1, \ldots, n,\ \forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geqslant 0},\ f_i(x) = x_i\, g_i(x)$$

with $g = (g_1, \ldots, g_n) : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{>0}$ a smooth order-preserving map.

I am interested in solving the equation

$$f(x) = b$$

for the unknown $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geqslant 0}$ given $b = (b_1, \ldots, b_n) \in \mathbb{R}^n_{\geqslant 0}$.

This equation is equivalent to the fixed-point equation

$$F(x) = x$$

where $F = (F_1, \ldots, F_n)$ and $F_i(x) = b_i/g_i(x)$.

5 Satisfying the Hypotheses of the Question

I provide here a map $f : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$ which satisfies the hypotheses of the question. After all, it could be that g (resp. F) being order-preserving (resp. order-reversing) is not enough.

Let I be a finite subset of $\mathbb{Z}^n_{\geqslant 0}$ and let $a_\alpha \in \mathbb{R}_{\geqslant 0}$ for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in I$. The map $f = (f_1, \ldots, f_n) : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$ given by

$$f_i(x) = x_i + \sum_{\alpha \in I} \alpha_i\, a_\alpha\, x^\alpha\,,$$

where $i = 1, \ldots, n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geqslant 0}$, satisfies the hypotheses of the question.

See [2, Theorem 3.4] for the fact that f is a smooth bijection. A sufficient but usually restrictive condition for the convergence of the sequence $\big(x(k)\big)_{k \geqslant 0}$ is in [1, Theorem 5.3]. The condition is not restrictive, i.e. is always satisfied, if all elements of I have ℓ_1-norm equal to 2; see [1, Theorem 5.4].

6 An Analogous Question

The feature $f_i(x) = x_i\, g_i(x)$ and the manner I wish to exploit it are also in Question [3]. But in my understanding, it is not known for a fact that the equation in that question has a unique nonnegative solution.

References

[1] Gilles Gnacadja, *Fixed Points of Order-Reversing Maps in $\mathbb{R}^n_{>0}$ and Chemical Equilibrium*, Mathematical Methods in the Applied Sciences 30 (2007), no. 2, 201–211, http://dx.doi.org/10.1002/mma.782.

[2] Gilles Gnacadja, *Univalent Positive Polynomial Maps and the Equilibrium State of Chemical Networks of Reversible Binding Reactions*, Advances in Applied Mathematics 43 (2009), no. 4, 394–414, http://dx.doi.org/10.1016/j.aam.2009.05.001.

[3] Tomek Tarczynski, *Convergence of Iterative Algorithm*, http://mathoverflow.net/questions/14167/convergence-of-iterative-algorithm.