Fixed Points of Order-Reversing Maps in $\mathbb{R}^n_{>0}$ and Chemical Equilibrium

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Abstract

The problem of computing the equilibrium state of a reversible chemical reaction network has a natural interpretation as a fixed-point problem. Several authors have used fixed-point iterations in this context, yet there are no comprehensive investigations into the convergence of the algorithm. We address this void by studying the larger problem of the existence and uniqueness of fixed points, and the convergence of fixed-point iterations, for order-reversing maps in $\mathbb{R}^n_{>0}$. By using the Thompson metric, we are able to apply fixed-point theorems based on the Lipschitz condition and obtain upper bounds on judiciously defined approximation errors.

Key words. Fixed point, Thompson metric, Superhomogeneous map, Order-reversing map, Chemical equilibrium.

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1 Introduction

The equilibrium state of a reversible chemical reaction network governed by mass-action kinetics is described by a multidimensional polynomial system. The system can readily be converted into a fixed-point equation for a reciprocal-polynomial map. A number of authors have used this observation to calculate the equilibrium by fixed-point iterations. However, there are no comprehensive investigations into the convergence of the algorithm. We look into this issue by studying the larger problem of the existence and uniqueness of fixed points, and the convergence of fixed-point iterations, for order-reversing real positive multidimensional maps.

Generally speaking, in a multidimensional situation in which ordinary norms (e.g. $\ell^\infty, \ell^1, \ell^2$) are used, it is difficult to find a bound for the norm of the Jacobian matrix that is useful in applying common fixed-point theorems based on the Lipschitz condition. In the case of positive order-reversing maps, we find that the Thompson metric is more amenable. By employing it, we are able to apply Edelstein and Banach's theorems on those maps that are superhomogeneous. This leads to conditions that ensure the convergence of the fixed-point algorithm for chemical equilibrium, and also estimates of relative errors in the approximation.

Uniqueness and convergence can occur even when the conditions we require do not apply. It is hence desirable to have the results under weaker conditions. Toward that goal, we provide an alternate proof in a particular case. The approach is based on global injectivity of multidimensional functions, and it is our hope that it can be generalized to more complex situations.

2 Superhomogeneous Maps

$\mathbb{R}^n$ is naturally equipped with the partial order $\leqslant$.
For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we have by definition:

$x \leqslant y$ if $\forall i = 1, \ldots, n, x_i \leqslant y_i$;

$x \lneqq y$ if $x \leqslant y$ and $x \neq y$;

$x < y$ if $\forall i = 1, \ldots, n, x_i < y_i$.

For $u, v \in \mathbb{R}^n$, we denote $[u, v]$, $[u, v[$, $]u, v]$, $]u, v[$ and call intervals or rectangles the sets of elements x of $\mathbb{R}^n$ that satisfy $u \leqslant x \leqslant v$, $u \leqslant x < v$, $u < x \leqslant v$, $u < x < v$ respectively. The notations and terminology extend naturally if $u = -\infty$ or $v = \infty$.

We shall say that a subset E of $\mathbb{R}^n$ is conical if it contains the open line segment between the origin and every point of E, i.e. $]0, 1[\cdot E \subseteq E$, where $]0, 1[\cdot E = \{tx : x \in E \text{ and } t \in]0, 1[\}$. Of course, a positive cone is conical, but here we want to include bounded sets.

A map f from a conical subset E of $\mathbb{R}^n$ to $\mathbb{R}^{n'}$ is said to be

k-subhomogeneous if	$\forall\, t \in \,]0,1[\,, \forall\, x \in E,\ t^k f(x) \leqslant f(tx);$
strictly k-subhomogeneous if	$\forall\, t \in \,]0,1[\,, \forall\, x \in E,\ t^k f(x) \lneqq f(tx);$
strongly k-subhomogeneous if	$\forall\, t \in \,]0,1[\,, \forall\, x \in E,\ t^k f(x) < f(tx).$

'Superhomogeneous' replaces 'subhomogeneous' in the definitions above when the inequalities are reversed. We collect in the following proposition a few obvious observations on these properties.

Proposition 2.1. *Let f be a map from a conical subset E of $\mathbb{R}^n$ to $\mathbb{R}^{n'}$.*

1. *Suppose $f(E) \subseteq \mathbb{R}^{n'}_{>0}$ and let $b \in \mathbb{R}^{n'}_{>0}$. Then f is k-subhomogeneous if and only if so is $bf : x \mapsto (b_1 f_1(x), \ldots, b_{n'} f_{n'}(x))$. The other five analogous equivalences hold.*

2. *Suppose $f(E) \subseteq \mathbb{R}^{n'}_{>0}$. Then f is k-subhomogeneous if and only if $\dfrac{1}{f} : x \mapsto \left(\dfrac{1}{f_1(x)}, \ldots, \dfrac{1}{f_{n'}(x)}\right)$ is $(-k)$-superhomogeneous. The other five analogous equivalences hold.*

3. *Subhomogeneity and superhomogeneity properties, and their strong forms, apply to f if and only they apply to each of the n' components of f.*

3 Thompson Metric and Lipschitz Condition

The Thompson metric d on $\mathbb{R}^n_{>0}$ is given, for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$, by

$$d(x, y) = \max_{1 \leqslant i \leqslant n} |\log x_i - \log y_i| = \max_{1 \leqslant i \leqslant n} \left|\log \frac{x_i}{y_i}\right| .$$

To see that d is indeed a metric, consider the metric d_∞ on $\mathbb{R}^n$ derived from the ℓ^∞ norm $\|\cdot\|_\infty$, and the map $\log^{\times n} : \mathbb{R}^n_{>0} \to \mathbb{R}^n$ such that $\log^{\times n} x = (\log x_1, \ldots, \log x_n)$; we have $d = d_\infty \circ (\log^{\times n} \times \log^{\times n})$ and $\log^{\times n}$ is injective.

The following proposition collects a few properties of this metric, including a characterization that is closer to how Thompson [1, Lemma 2] originally introduced it.

Proposition 3.1.

1. *Let $\alpha \in \mathbb{R}^n_{>0}$. The map $x \mapsto (\alpha_1 x_1, \ldots, \alpha_n x_n)$ is an isometry on $(\mathbb{R}^n_{>0}, d)$.*

2. *The reciprocal map* $x \mapsto \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$ *is an isometry on* $(\mathbb{R}^n_{>0}, d)$.

3. *Let* $x, y \in \mathbb{R}^n_{>0}$. *We have* $d(x, y) = \log \mu(x, y)$, *where* $\mu(x, y)$ *is characterized by* $[\mu(x, y), \infty[= \left\{\lambda \geqslant 1 : \frac{1}{\lambda} y \leqslant x \leqslant \lambda y\right\}$.
   If $x \geqslant y$, *then* $\mu(x, y) = \max_{1 \leqslant i \leqslant n} \frac{x_i}{y_i}$; *and*
   for $x', y' \in [y, x]$, *we have* $\mu(x', y') \leqslant \mu(x, y)$ *and* $d(x', y') \leqslant d(x, y)$.

4. *The metrics* d *and* d_∞ *are locally equivalent, and hence topologically equivalent on* $\mathbb{R}^n_{>0}$: *for every* $c, b \in \mathbb{R}^n_{>0}$ *with* $c \leqslant b$, *we have*
   $(\min c)\, d \leqslant d_\infty \leqslant (\max b)\, d$ *on* $[c, b] \times [c, b]$. *Also, with* $d_{\infty,b}$ *defined by* $d_{\infty,b}(x, y) = \max_{1 \leqslant i \leqslant n} \left|\frac{x_i - y_i}{b_i}\right|$, *we have* $d_{\infty,b} \leqslant d \leqslant \mu(b, c)\, d_{\infty,b}$ *on* $[c, b] \times [c, b]$. □

We omit the trivial proofs of these properties. Let's just mention that the inequalities in the fourth statement are obtained effortlessly from the fact that, by the Mean Value Theorem, for every $u, v \in \mathbb{R}_{>0}$, we have $v - u = (\log v - \log u)w$ for some w between u and v. The metric $d_{\infty,b}$ will be used later (Theorem 4.2) in error estimation. Next we provide results on the Lipschitz condition with respect to the Thompson metric.

Proposition 3.2. *Let* E *be a conical subset in* $\mathbb{R}^n_{>0}$, *let* $k \in \mathbb{R}_{>0}$, *and consider a map* $f : E \to \mathbb{R}_{>0}$. *Assume both* E *and* $\mathbb{R}_{>0}$ *are equipped with the Thompson metric.*

- *If* f *is* k*-Lipschitz (resp. strictly* k*-Lipschitz), then* f *is both* k*-subhomogeneous and* $(-k)$*-superhomogeneous (resp. strongly* k*-subhomogeneous and strongly* $(-k)$*-superhomogeneous).*

- *If* f *is order-preserving, then* f *is* k*-Lipschitz (resp. strictly* k*-Lipschitz) if and only if* f *is* k*-subhomogeneous (resp. strongly* k*-subhomogeneous).*
  If f *is order-reversing, then* f *is* k*-Lipschitz (resp. strictly* k*-Lipschitz) if and only if* f *is* $(-k)$*-superhomogeneous (resp. strongly* $(-k)$*-superhomogeneous).*

Parts of this proposition with homogeneity in lieu of subhomogeneity and superhomogeneity are in Nussbaum [2, Proposition 1.5].

Proof. It is sufficient to prove the stronger statements. Suppose f is strictly k-Lipschitz. Let $x \in E$ and $t \in]0, 1[$. We have $d(f(x), f(tx)) < k\, d(x, tx)$, and so $\mu(f(x), f(tx)) < (\mu(x, tx))^k$. But $\mu(x, tx) = t^{-1}$, so $\mu(f(x), f(tx)) <$

t^{-k}, whence $f(x) < t^{-k}f(tx)$ and $f(tx) < t^{-k}f(x)$. So f is strongly k-subhomogeneous and strongly $(-k)$-superhomogeneous. For the converse, suppose f is order-preserving and strongly k-subhomogeneous. Let $x, y \in E$ with $x \neq y$. Let $\lambda = \mu(x, y)$. We have $\lambda > 1$ and $\lambda^{-1}x \leqslant y$. Then, $\lambda^{-k}f(x) < f(\lambda^{-1}x) \leqslant f(y)$, so $f(x) < \lambda^k f(y)$. Likewise, $f(y) < \lambda^k f(x)$. So $\mu(f(x), f(y)) < \lambda^k = (\mu(x, y))^k$, whence $d(f(x), f(y)) < k\, d(x, y)$. So f is strictly k-Lipschitz. If f is order-reversing and strongly $(-k)$-superhomogeneous, then $1/f$ is order-preserving and strongly k-subhomogeneous, and therefore $1/f$ is strictly k-Lipschitz. Then by Proposition 3.1, part 2, f is strictly k-Lipschitz. $\square$

4 Fixed Points

Like the Banach Fixed Point Theorem, the Edelstein Fixed Point Theorem (see [3, Theorem 1] or [4, Theorem 2.6]) asserts the existence and uniqueness of a fixed point and the convergence of all iterations to the fixed point. It has a stronger requirement on the metric space (compact instead of complete) and a weaker requirement on the map (contractive, i.e. strictly 1-Lipschitz, instead of contraction). We use it for the following result.

Theorem 4.1. *Consider an order-reversing map $f : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{>0}$ such that for some $m_0 \in \mathbb{Z}_{\geqslant 0}$, f is strongly (-1)-superhomogeneous on $]0, f^{2m_0+1}(0)]$. Then f has a unique fixed point ω, and for every $x \in \mathbb{R}^n_{\geqslant 0}$, $f^m(x) \xrightarrow[m\to\infty]{} \omega$.*

Proof. By Proposition 3.2 and part 3 of Proposition 2.1, f contractive with respect to the Thompson metric on $]0, f^{2m_0+1}(0)]$, and hence on $K = [f^{2m_0}(0), f^{2m_0+1}(0)]$. Furthermore, K is nonempty compact, and $f(K) \subseteq K$. So by the Edelstein Fixed Point Theorem, f has a unique fixed point ω in K, and for every $x \in K$, $f^m(x) \xrightarrow[m\to\infty]{} \omega$. This statement holds with $\mathbb{R}^n_{\geqslant 0}$ in lieu of K because $f^{2m_0+1}(\mathbb{R}^n_{\geqslant 0}) \subseteq K$. $\square$

Theorem 4.1 and its proof implicitly use the fact that the Thompson topology is equivalent to the ordinary topology (Proposition 3.1, part 4).

While Theorem 4.1 is obtained here through techniques based on the Lipschitz condition in a metric space, we note that Guo [5, Theorem 2.1] has used another approach which, in our circumstances, goes as follows. The sequences $(f^{2m}(0))_{m\geqslant 0}$ and $(f^{2m+1}(0))_{m\geqslant 0}$ are increasing and decreasing, and converge to limit points ω and ω' respectively. We have $\omega \leqslant \omega'$, $f(\omega) = \omega'$, $f(\omega') = \omega$, and $\bigcap_{m\geqslant 0} f^m\left(\mathbb{R}^n_{\geqslant 0}\right) = [\omega, \omega']$. The essential point is to prove that $\omega = \omega'$. Let $\lambda \geqslant \mu(\omega, \omega')$, and suppose that $\lambda > 1$. Then $\lambda^{-1}\omega' \leqslant \omega$ and, because f is order-reversing, $\omega' = f(\omega) \leqslant f(\lambda^{-1}\omega')$. Now because f is strongly (-1)-superhomogeneous, we have $f(\lambda^{-1}\omega') < \lambda f(\omega') = \lambda\omega$. So $\omega' < \lambda\omega$, and this implies $\lambda > \mu(\omega, \omega')$. By contrapositive, $\mu(\omega, \omega') = 1$, whence $\omega = \omega'$. We revisit this approach in the latter part of section 5.

Theorem 4.1 is of course valid if f is $(-k)$-superhomogeneous on $]0, f(0)]$ with $0 < k < 1$. We can use the fact that f is a k-contraction with respect to the Thompson metric to bound the error in approximating the fixed point with iterates. We obtain the following theorem by combining the inequalities $d(f^m(x), f^{m+1}(x)) \leqslant k^m d(x, f(x))$ and $d(f^m(x), \omega) \leqslant \dfrac{k^m}{1-k} d(x, f(x))$ (see for instance [4, Theorem 2.2]) with inequalities from Proposition 3.1.

Theorem 4.2. *Consider an order-reversing map $f : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{>0}$. Let $b = f(0)$, $c = f(b) = f^2(0)$, and let $k \in \mathbb{R}$ with $0 < k < 1$. Suppose f is $(-k)$-superhomogeneous on $]0, b]$, and let ω be the unique fixed point of f. For $x \in \mathbb{R}^n_{\geqslant 0}$ and $m \in \mathbb{Z}_{\geqslant 0}$, let*

$$\varepsilon_m(x) = \max_{1 \leqslant i \leqslant n} \left| \frac{(f^m(x))_i - \omega_i}{b_i} \right| \quad \text{and} \quad \widetilde{\varepsilon}_m(x) = \max_{1 \leqslant i \leqslant n} \left| \frac{(f^m(x))_i - (f^{m+1}(x))_i}{b_i} \right| .$$

Let $\rho = d(b, c) = \log \max_{1 \leqslant i \leqslant n} \left(\dfrac{b_i}{c_i} \right)$. For every $x \in [c, b]$ and $m \in \mathbb{Z}_{\geqslant 0}$,

$$\varepsilon_m(x) \leqslant \frac{\rho\, k^m}{1-k} \qquad \text{and} \qquad \widetilde{\varepsilon}_m(x) \leqslant \rho\, k^m .$$

□

Remark 4.3. $\varepsilon_m(x)$ and $\widetilde{\varepsilon}_m(x)$ are judicious choices for relative error estimation because the absolute error for each component of f is normalized by the maximum value the component can assume. Theorem 4.2 provides bounds for these errors, but even under the weaker assumptions of Theorem 4.1, the sequences $(\varepsilon_m(x))_{m \geqslant 0}$ and $(\widetilde{\varepsilon}_m(x))_{m \geqslant 0}$ converge to zero. If $x \in [c, b]$ satisfies $f^2(x) \leqslant x$, e.g. if $x = b$, then the sequence $(f^{2m}(x))_{m \geqslant 0}$ decreases to ω, the sequence $(f^{2m+1}(x))_{m \geqslant 0}$ increases to ω, we have $\varepsilon_m(x) \leqslant \widetilde{\varepsilon}_m(x)$, and the sequence $(\widetilde{\varepsilon}_m(x))_{m \geqslant 0}$ decreases to 0. In this case, imposing an upper bound on $\widetilde{\varepsilon}_m(x)$ is a well-suited iteration stopping condition. □

5 Example: Reciprocal Polynomials

Proposition 5.1. *Let I be a nonempty finite subset of $\mathbb{Z}^n_{\geqslant 0}\backslash\{0\}$, $a_0 \in \mathbb{R}_{>0}$, $a_\alpha \in \mathbb{R}_{>0}$ for each $\alpha \in I$, $b \in \mathbb{R}^n_{>0}$, $k = \dfrac{\sum_{\alpha\in I} |\alpha|\, a_\alpha\, b^\alpha}{a_0 + \sum_{\alpha\in I} a_\alpha\, b^\alpha}$, and $b' \in \mathbb{R}_{>0}$. Consider the polynomial function $P : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}_{>0}$ given by $P(x) = a_0 + \sum_{\alpha\in I} a_\alpha\, x^\alpha$ and the function $f : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}_{>0}$ given by $f(x) = \dfrac{b'}{P(x)}$.*
If $\sum_{\alpha\in I, |\alpha|\geqslant 2} (|\alpha| - 1)\, a_\alpha\, b^\alpha \leqslant a_0$, or equivalently $0 < k \leqslant 1$, then P is strongly k-subhomogeneous on $]0, b]$, f is strongly $(-k)$-superhomogeneous on $]0, b]$, and P and f are strictly k-Lipschitz on $]0, b]$ with respect to the Thompson metric.

We clarify two notations: $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Proof. Let $\varphi(x, t) = t^k P(x) - P(tx)$. We have:

$$\begin{aligned}
\varphi(x,t) &= t^k \left(a_0 + \sum_{\alpha\in I} a_\alpha\, x^\alpha \right) - a_0 - \sum_{\alpha\in I} t^{|\alpha|} a_\alpha\, x^\alpha\,, \\
\frac{\partial\varphi}{\partial t}(x,t) &= k t^{k-1} \left(a_0 + \sum_{\alpha\in I} a_\alpha\, x^\alpha \right) - \sum_{\alpha\in I} |\alpha| t^{|\alpha|-1} a_\alpha\, x^\alpha\,.
\end{aligned}$$

Suppose $k = 1$. Then

$$\frac{\partial\varphi}{\partial t}(x,t) = a_0 + \sum_{\alpha\in I, |\alpha|\geqslant 2} a_\alpha\, x^\alpha - \sum_{\alpha\in I, |\alpha|\geqslant 2} |\alpha| t^{|\alpha|-1} a_\alpha\, x^\alpha\,.$$

If $a_\alpha = 0$ when $|\alpha| \geqslant 2$, then $\dfrac{\partial\varphi}{\partial t}(x,t) = a_0 > 0$. If not, then if $t \in]0, 1[$ and $x \in]0, b]$, we have

$$\begin{aligned}
\frac{\partial\varphi}{\partial t}(x,t) &> a_0 + \sum_{\alpha\in I, |\alpha|\geqslant 2} a_\alpha\, x^\alpha - \sum_{\alpha\in I, |\alpha|\geqslant 2} |\alpha| a_\alpha\, x^\alpha \\
&= a_0 - \sum_{\alpha\in I, |\alpha|\geqslant 2} (|\alpha| - 1) a_\alpha\, x^\alpha \\
&\geqslant a_0 - \sum_{\alpha\in I, |\alpha|\geqslant 2} (|\alpha| - 1) a_\alpha\, b^\alpha \\
&= 0\,.
\end{aligned}$$

Now suppose $0 < k < 1$. Then for $t \in]0,1[$ and $x \in]0,b]$, we have

$$\begin{aligned} t^{1-k}\frac{\partial\varphi}{\partial t}(x,t) &= k\left(a_0 + \sum_{\alpha\in I} a_\alpha\, x^\alpha\right) - \sum_{\alpha\in I} |\alpha| t^{|\alpha|-k} a_\alpha\, x^\alpha \\ &> k\left(a_0 + \sum_{\alpha\in I} a_\alpha\, x^\alpha\right) - \sum_{\alpha\in I} |\alpha| a_\alpha\, x^\alpha \\ &= ka_0 - \sum_{\alpha\in I} (|\alpha| - k) a_\alpha\, x^\alpha \\ &\geqslant ka_0 - \sum_{\alpha\in I} (|\alpha| - k) a_\alpha\, b^\alpha \\ &= k\left(a_0 + \sum_{\alpha\in I} a_\alpha\, b^\alpha\right) - \sum_{\alpha\in I} |\alpha| a_\alpha\, b^\alpha \\ &= 0\,. \end{aligned}$$

So we have obtained that if $0 < k \leqslant 1$, then $\dfrac{\partial\varphi}{\partial t}(x,t) > 0$ for $t \in]0,1[$ and $x \in]0,b]$. It follows that $\varphi(x,t) < \varphi(x,1) = 0$, i.e. $t^k P(x) < P(tx)$. So P is strongly k-subhomogeneous on $]0,b]$. Then by Proposition 2.1, f is strongly $(-k)$-superhomogeneous on $]0,b]$. Finally by Proposition 3.2, P and f are strictly k-Lipschitz with respect to the Thompson metric on $]0,b]$. □

Note that if all $\alpha \in I$ satisfy $|\alpha| = 1$, then the condition $0 < k < 1$ is satisfied independently of b. So the following result is a noteworthy particular instance of Proposition 5.1.

Proposition 5.2. *Let $a_0, b \in \mathbb{R}_{>0}$, $a \in \mathbb{R}^n_{\geqslant 0}\backslash\{0\}$, $k = \dfrac{a\cdot b}{a_0 + a\cdot b}$, and consider the reciprocal affine function $f : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}_{>0}$ given by*

$$f(x) = \frac{1}{a_0 + a\cdot x} = \frac{1}{a_0 + a_1x_1 + \cdots + a_nx_n}\,.$$

We have $0 < k < 1$, and with respect to the Thompson metric, f is contractive on $\mathbb{R}^n_{>0}$ and strictly k-Lipschitz on $]0,b]$. (So is the affine function $x \mapsto a_0 + a\cdot x$.) □

It is interesting to contrast this result for $n = 1$ with the situation where the real line is equipped with its ordinary metric. For instance, let $a \in \mathbb{R}_{>1}$ and $b = 1 - \dfrac{1}{a}$, $k = 1 - \dfrac{1}{a}$. With respect to the Thompson metric, the function $f : x \mapsto \dfrac{a}{1+ax}$ is contractive on $\mathbb{R}_{>0}$ and is strictly k-Lipschitz on $]0,b]$.

But for $x, y \in [0, b]$ with $x \neq y$, we have $|f(x) - f(y)| > |x - y|$.

We now combine Proposition 5.1 and the results of section 4 in the following Theorem.

Theorem 5.3. *Let $I_1, \ldots, I_n$ be nonempty finite subsets of $\mathbb{Z}^n_{\geqslant 0} \backslash \{0\}$, and $b \in \mathbb{R}^n_{>0}$. For $i = 1, \ldots, n$ and $\alpha \in I_i$, let $a_{i\alpha} \in \mathbb{R}_{>0}$. Let $k_i = \dfrac{\sum_{\alpha \in I_i} |\alpha|\, a_{i\alpha}\, b^\alpha}{1 + \sum_{\alpha \in I_i} a_{i\alpha}\, b^\alpha}$, $k = \max(k_1, \ldots, k_n)$, and $\rho = \log\left(1 + \max\limits_{1 \leqslant i \leqslant n} \sum_{\alpha \in I_i} a_{i\alpha}\, b^\alpha\right)$. Consider the function $f = (f_1, \ldots, f_n) : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{>0}$ given by $f_i(x) = \dfrac{b_i}{1 + \sum_{\alpha \in I_i} a_{i\alpha}\, x^\alpha}$. Suppose that $\sum_{\alpha \in I_i, |\alpha| \geqslant 2} (|\alpha| - 1)\, a_{i\alpha}\, b^\alpha \leqslant 1$ for every $i = 1, \ldots, n$, or equivalently that $0 < k \leqslant 1$. Then f has a unique fixed point ω, and for every $x \in \mathbb{R}^n_{\geqslant 0}$, $f^m(x) \xrightarrow[m \to \infty]{} \omega$; the sequence $(f^{2m}(b))_{m \geqslant 0}$ decreases to ω, the sequence $(f^{2m+1}(b))_{m \geqslant 0}$ increases to ω, $\varepsilon_m(b) \leqslant \widetilde{\varepsilon}_m(b)$, and the sequence $(\widetilde{\varepsilon}_m(b))_{m \geqslant 0}$ decreases to 0. If in fact $\sum_{\alpha \in I_i, |\alpha| \geqslant 2} (|\alpha| - 1)\, a_{i\alpha}\, b^\alpha < 1$ for every $i = 1, \ldots, n$, or equivalently $0 < k < 1$, then $\varepsilon_m(x) \leqslant \dfrac{\rho\, k^m}{1 - k}$ and $\widetilde{\varepsilon}_m(x) \leqslant \rho\, k^m$ for $x \in [f(b), b]$, and $\varepsilon_m(b) \leqslant \widetilde{\varepsilon}_m(b) \leqslant \rho\, k^m$.*

Proof. First suppose $k \leqslant 1$. For each $i = 1, \ldots, n$, $k_i \leqslant 1$, so by Proposition 5.1, f_i is strongly $(-k)$-superhomogeneous on $]0, b]$. Then, by part 3 of Proposition 2.1, f is strongly $(-k)$-superhomogeneous on $]0, b]$. Since $b = f(0)$, Theorem 4.1 implies that f has a unique fixed point ω, and for every $x \in \mathbb{R}^n_{\geqslant 0}$, $f^m(x) \xrightarrow[m \to \infty]{} \omega$. Now if $k < 1$, Theorem 4.2 provides the bounds on the relative errors. □

In the following result, we highlight what the theorem says if all $\alpha \in I$ satisfy $|\alpha| = 1$.

Theorem 5.4. *Let $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ be a nonnegative matrix with no zero rows, $b \in \mathbb{R}^n_{>0}$, and $f = (f_1, \ldots, f_n) : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{>0}$ given by*

$$f_i(x) = \frac{b_i}{1 + a_{i1}x_1 + \ldots + a_{in}x_n} \,.$$

In the notations of Theorem 5.3, we have $0 < k < 1$ for any choice of b;

$$k = \frac{\|Ab\|_\infty}{1 + \|Ab\|_\infty} \qquad \text{and} \qquad \rho = \log(1 + \|Ab\|_\infty) \,.$$

□

Remark 5.5. The condition in Theorem 5.3 is sufficient but not necessary for convergence. Consider for instance the function $f : \mathbb{R}_{\geqslant 0} \to]0,1]$ given by $f(x) = \dfrac{1}{1+x^p}$. Theorem 5.3 requires $p \leqslant 2$. However, for $p = 3$, with $k = \dfrac{2}{3}\sqrt[3]{2} < 1$, f is k-Lipschitz (a contraction) on $\mathbb{R}_{\geqslant 0}$ with respect to the ordinary metric. We leave the verification of this as an exercise. $\square$

Application to Chemical Equilibrium

We consider a reversible chemical reaction network involving a finite number of chemical species X_σ, where $\sigma = (\sigma_1, \ldots, \sigma_n) \in I$; I is a finite subset of $\mathbb{Z}^n_{\geqslant 0} \backslash \{0\}$ containing $e_1, \ldots, e_n$, where e_i has 1 in position i and 0 elsewhere. Let $X_i = X_{e_i}$. $X_1, \ldots, X_n$ are the simple species, and the reactions that can occur are

$$\sigma_1 X_1 + \cdots + \sigma_n X_n \rightleftharpoons X_\sigma \,.$$

We assume that the network is governed by mass-action kinetics. Then by the Deficiency-Zero Theorem (see for instance Feinberg [6, Theorem 4.1]), this system admits a unique equilibrium state for any choice of total concentrations of simple species. We are interested in finding the equilibrium concentrations y_σ of X_σ. Let $x = (x_1, \ldots, x_n)$ be the vector of equilibrium concentrations of simple species (so $x_i = y_{e_i}$). As a consequence of the mass-action kinetics assumption, we have the binding constants K_σ such that $y_\sigma = K_\sigma\, x^\sigma = K_\sigma\, x_1^{\sigma_1} \cdots x_n^{\sigma_n}$. So if x is known, it is easy to calculate the other equilibrium concentrations. We work with molar concentrations and denote b_i the total (free and bound) concentration of X_i. This is a time-invariant of the system. In particular we have

$$b_i = \sum_{\sigma \in I} \sigma_i\, y_\sigma = \sum_{\sigma \in I} \sigma_i\, K_\sigma\, x^\sigma \,.$$

Let $I_i = \{\sigma - e_i : \sigma \in I, \sigma \geqslant e_i, \sigma \neq e_i\}$. The complex species that actually have X_i as a constituent are $X_{e_i+\alpha}$ with $\alpha \in I_i$. We have

$$b_i = x_i \left(1 + \sum_{\alpha \in I_i} (1 + \alpha_i)\, K_{e_i+\alpha}\, x^\alpha \right) .$$

With $a_{i\alpha} = (1+\alpha_i)\, K_{e_i+\alpha}$, this amounts to $f(x) = x$ in the notation of Theorem 5.3. Hence the theorem shows that the vector of equilibrium concentrations of simple species may be efficiently calculated as the fixed point of f by iterating f starting at the vector b of total concentrations of simple species.

We have implemented this algorithm in a computer program in Java. The program reads reaction network information from an XML document and validates it against an XML schema which we have designed for the specification of reversible chemical reaction networks. We actually employed the program to calculate the equilibrium state of several actual biochemical reaction networks. Agreement with experimental data when available was very good. We achieved values of $\widetilde{\varepsilon}_m(b)$ smaller than 10^{-9} in 2 to 25 iterations. We observed nonconvergence only after letting binding constants and/or total concentrations assume unrealistically large values.

History

We discovered this method for calculating the equilibrium state in 2005 while working on a biochemical system of interest in therapeutic research in inflammation. Then from Pedro Mendes [7], we learned of the work of Kuzmič [8]. This led us to several references, starting with Perrin [9] where the algorithm is used, but not proved to converge. In Perrin and Sayce [10], the authors employ the algorithm in a modified fashion; instead of f, they use the map that is the midpoint on the 'multiplicative' homotopy path between f and the identity map. The algorithm is presented in a more systematic fashion in Storer, Cornish and Bowden [11], and this is used in Barshop, Wrenn and Frieden [12]. We found in Kuzmic [8] the first attempt to investigate convergence properties. But the author asserts that this cannot be done in a multidimensional situation and works out at the one-dimensional case. To our knowledge, we have presented in this paper the first convergence results on the multidimensional algorithm. We note that other methods of calculating chemical equilibrium have been studied. In particular, Meintjes and Morgan [13] present a systematic approach to reduce the polynomial system, and then numerically solve the reduced system by homotopy continuation or Newton's method.

Alternate Proof

It is desirable to have other means of investigating existence, uniqueness and convergence, since these can occur even if the conditions of Theorem 5.3 are not satisfied (see Remark 5.5), and proving that a map in $\mathbb{R}^n$ is a contraction with respect to the ordinary metrics is not necessarily easy. One approach is, given a continuous order-reversing map $f : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{>0}$ for which Lipschitz or superhomogeneity properties are not known, to show that the points $\omega = \lim_{m\to\infty} f^{2m}(0) = \sup_m f^{2m}(0)$ and $\omega' = \lim_{m\to\infty} f^{2m+1}(0) = \inf_m f^{2m+1}(0)$ are equal. (See the discussion following the proof of Theorem 4.1.) ω and ω' form an orbit of fixed points of f^2 and $\omega \leqslant \omega'$. So it is enough to show that f^2 has no distinct comparable fixed points, or better, that f^2 has at most one fixed point. We show a way to do that for f as in Theorem 5.4, with the matrix A symmetric. This is less general, but it does apply to chemical reaction networks with only binary bindings, and the goal is to propose a method that might be extended to more complex situations.

Proof (Alternate proof of Theorem 5.4 with A symmetric).
Let the map $g = (g_1, \ldots, g_n) : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$ be defined by

$$g_i(x) = x_i + \sum_{j=1}^{n} \frac{a_{ij} b_j x_i}{1 + \sum_{k=1}^{n} a_{jk} x_k} .$$

We have $f^2(x) = x \Leftrightarrow g(x) = b$. We show that f^2 has at most one fixed point by proving that g is injective. By a result of Gale and Nikaidô [14, Theorem 4 and Remark 4.3] on global injectivity, it suffices to show that the Jacobian matrix $Jg(x)$ of g at any $x \in \mathbb{R}^n_{\geqslant 0}$ is a P-matrix, which we do by showing that $Jg(x)$ has positive diagonal entries and is strictly diagonally dominant by column; see for instance Berman and Plemmons [15, Theorem 6.2.3, conditions (A_1) and (M_{35})] for these matrix notions. We have

$$\begin{aligned}
\partial_i g_i(x) &= 1 + \sum_{j=1}^{n} a_{ij} b_j \frac{1 + \sum_{k \neq i, k=1}^{n} a_{jk} x_k}{\left(1 + \sum_{k=1}^{n} a_{jk} x_k\right)^2} \\
&= 1 + \sum_{j=1}^{n} \frac{a_{ij} b_j}{\left(1 + \sum_{k=1}^{n} a_{jk} x_k\right)^2} + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{n} \sum_{j=1}^{n} \frac{a_{ij} a_{j\ell} b_j x_\ell}{\left(1 + \sum_{k=1}^{n} a_{jk} x_k\right)^2}
\end{aligned}$$

and, if $i \neq \ell$,

$$\partial_\ell g_i(x) = -\sum_{j=1}^{n} \frac{a_{ij} a_{j\ell} b_j x_i}{\left(1 + \sum_{k=1}^{n} a_{jk} x_k\right)^2} .$$

So

$$\sum_{\ell=1}^{n} \partial_i g_\ell(x) = 1 + \sum_{j=1}^{n} \frac{a_{ij} b_j}{\left(1 + \sum_{k=1}^{n} a_{jk} x_k\right)^2} + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{n} \sum_{j=1}^{n} \frac{(a_{ij} a_{j\ell} - a_{ji} a_{\ell j}) b_j x_\ell}{\left(1 + \sum_{k=1}^{n} a_{jk} x_k\right)^2} \,.$$

Since A is symmetric, we have

$$\sum_{\ell=1}^{n} \partial_i g_\ell(x) = 1 + \sum_{j=1}^{n} \frac{a_{ij} b_j}{\left(1 + \sum_{k=1}^{n} a_{jk} x_k\right)^2} > 0 \,.$$

Since $\partial_i g_i(x) > 0$ and $\partial_i g_\ell(x) \leqslant 0$ for $i \neq \ell$, we have proved that $Jg(x)$ has the desired properties. □

6 Conclusion

We provided conditions for order-reversing maps in $\mathbb{R}^n_{>0}$ to have desirable properties for the purpose of computing their fixed points, and applied the results to the computation of chemical equilibrium. The key requirement is superhomogeneity. It can however be too much to require, and we intend to look for weaker requirements, for example through global injectivity.

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