Equilibria Exist in Compact Convex Forward-Invariant Sets

http://mathoverflow.net/questions/68174/equilibria-exist-in-compact-convex-forward-invariant-sets
http://gillesgnacadja.wordpress.com/2011/06/18/equilibria-exist-in-compact-convex-forward-invariant-sets

Gilles Gnacadja
http://math.gillesgnacadja.info/
gilles.gnacadja@gmail.com

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Theorem. *Consider a continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ and suppose that the autonomous dynamical system $\dot{x} = f(x)$ has a semiflow $\varphi : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$. Let $K \subseteq \mathbb{R}^n$. If K is nonempty, compact, convex and forward-invariant, then K contains an equilibrium of the dynamical system, i.e. a zero of the map f.*

According to a reliable source, the above theorem is a standard result everyone uses in dynamical systems without proof. I propose a proof in this document. With $\mathrm{Zero}(f)$ denoting the set of zeros of f, the result is that $K \cap \mathrm{Zero}(f) \neq \varnothing$ for any nonempty, compact, convex, forward-invariant $K \subset \mathbb{R}^n$.

The semiflow φ satisfies the following properties.

- The map $\varphi : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.
- For every $a \in \mathbb{R}^n$, the map $\varphi(-, a) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n, t \mapsto \varphi(t, a)$ is of class C^1 and is the solution trajectory originating at a, i.e. $\varphi(0, a) = a$ and $\forall\, t \in \mathbb{R}_{\geqslant 0}\,,\ \dfrac{\partial \varphi}{\partial t}(t, a) = f\big(\varphi(t, a)\big)$.
- $\forall\, t, t' \in \mathbb{R}_{\geqslant 0}\,,\ \forall\, a \in \mathbb{R}^n\,,\ \varphi(t + t', a) = \varphi\big(t', \varphi(t, a)\big)$.

For $a \in \mathbb{R}^n$, we have

$$f(a) = 0 \iff \forall\, t \in \mathbb{R}_{\geqslant 0}\,,\ \varphi(t, a) = a\ . \tag{1}$$

With $\Phi(t)$ denoting the set of fixed points of $\varphi(t,-)$ for each $t \in \mathbb{R}_{\geqslant 0}$, Property (1) is equivalent to

$$\mathrm{Zero}(f) \;=\; \bigcap_{t \in \mathbb{R}_{\geqslant 0}} \Phi(t)\;. \tag{2}$$

Because $\varphi(t,-)$ is continuous for each $t \in \mathbb{R}_{\geqslant 0}$ and $\mathbb{Q}_{\geqslant 0}$ is dense in $\mathbb{R}_{\geqslant 0}$, we also have

$$\mathrm{Zero}(f) \;=\; \bigcap_{t \in \mathbb{Q}_{\geqslant 0}} \Phi(t)\;. \tag{3}$$

A straightforward inductive reasoning shows that

$$\forall\, t \in \mathbb{R}_{\geqslant 0}\,,\; \forall\, n \in \mathbb{Z}_{\geqslant 0}\,,\; \Phi(t) \subseteq \Phi(nt)\;. \tag{4}$$

It then results that

$$\forall\, t_1, t_2 \in \mathbb{Q}_{\geqslant 0}\,,\; \exists\, t \in \mathbb{Q}_{\geqslant 0} \;:\; \Phi(t) \subseteq \Phi(t_1) \cap \Phi(t_2)\;. \tag{5}$$

Indeed, with $i \in \{1,2\}$, let $t_i \in \mathbb{Q}_{\geqslant 0}$, and let $p_i \in \mathbb{Z}_{\geqslant 0}$ and $q_i \in \mathbb{Z}_{>0}$ such that $t_i = p_i/q_i$. Then let $n_1 = p_1 q_2$, $n_2 = p_2 q_1$, and $t = 1/(q_1 q_2)$. We have $t \in \mathbb{Q}_{\geqslant 0}$, $n_i \in \mathbb{Z}_{\geqslant 0}$ and $t_i = n_i t$. By Property (4), $\Phi(t) \subseteq \Phi(t_i)$.

On another hand, we have

$$\forall\, t \in \mathbb{R}_{\geqslant 0}\,,\; K \cap \Phi(t) \neq \varnothing\;. \tag{6}$$

Indeed, let $t \in \mathbb{R}_{\geqslant 0}$. Because K is forward-invariant, the (continuous) map $\varphi(t,-) : \mathbb{R}^n \to \mathbb{R}^n$ restricts to a continuous map $K \to K$. And because K is compact and convex, the Brouwer Fixed Point Theorem implies that $\varphi(t,-)$ has a fixed point in K.

Properties (6) and (5) together say that the family $\left\{K \cap \Phi(t)\right\}_{t \in \mathbb{Q}_{\geqslant 0}}$ is a filter basis and imply that the family has the finite intersection property: for every finite $T \subset \mathbb{Q}_{\geqslant 0}$, $\bigcap_{t \in T} \big(K \cap \Phi(t)\big) \neq \varnothing$. Furthermore, for every $t \in \mathbb{R}_{\geqslant 0}$, $K \cap \Phi(t)$ is a closed subset of K because $\Phi(t)$ is a closed subset of $\mathbb{R}^n$. Since K is compact, we have

$$\varnothing \;\neq\; \bigcap_{t \in \mathbb{Q}_{\geqslant 0}} \big(K \cap \Phi(t)\big) \;=\; K \cap \bigcap_{t \in \mathbb{Q}_{\geqslant 0}} \Phi(t) \;=\; K \cap \mathrm{Zero}(f)\;. \tag{7}$$

The proof is complete.