Asymptotic Equidistribution of Congruence Classes with respect to the Convolution Iterates of a Probability Vector

Supplementary Article for

A Mathematical Model for Projecting the Replenishment of Compounds in a Sample Bank

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This supplementary article strengthens the findings published in Gnacadja [1] so as to support the proof of Result 5 in the main article.

Let $f = \big(f(0), \ldots, f(\ell)\big)$ be a probability vector. For $n \in \mathbb{Z}_{\geqslant 0}$, let f^{*n} denote the n-fold convolution of f. This is a probability vector over the numbers $0, \ldots, n\ell$ representing the n-fold repetition of the process represented by f.

For $d \in \mathbb{Z}_{\geqslant 1}$ and $r = 0, \ldots, d-1$, let $\varphi(f, n, d, r)$ denote the probability that a component number in f^{*n} is congruent to r modulo d. By definition, we have

$$\varphi(f,n,d,r) \;=\; \sum_{\substack{0\leqslant k\leqslant n\ell \\ k\equiv r \bmod d}} f^{*n}(k) \;=\; \sum_{q=0}^{\operatorname{floor}((n\ell-r)/d)} f^{*n}(r+d\,q)\ .$$

We assemble these probabilities into a probability vector as follows.

$$\varphi(f,n,d) \;:=\; \big(\varphi(f,n,d,0), \ldots, \varphi(f,n,d,d-1)\big)$$

Observe that $\varphi(f,n,1) = \big(\varphi(f,n,1,0)\big) = (1)$. Suppose $d \geqslant 2$ and let

$$\omega_d \;:=\; \exp\left(\frac{2\pi\mathrm{i}}{d}\right) \qquad \text{and} \qquad P_{f,d}(X) \;:=\; \sum_{r=0}^{d-1} \varphi(f,1,d,r) X^r\ ,$$

and then

$$\gamma(f,d) \;:=\; \max_{1\leqslant r\leqslant d-1} \left| P_{f,d}\left(\omega_d^r\right) \right|\ .$$

Also, let

$$u_d \ := \ \frac{1}{d}\,(\underbrace{1,\ldots,1}_{d}) \quad \text{and} \quad e_d \ := \ (1,\underbrace{0,\ldots,0}_{d-1})\ .$$

Theorem 1. *Let $f = \big(f(0),\ldots,f(\ell)\big)$ be a positive probability vector with $\ell \geqslant 1$, and let $d \in \mathbb{Z}_{\geqslant 2}$. We have*

$$\gamma(f,d) \ < \ 1 \tag{1}$$

and

$$\forall\, n \in \mathbb{Z}_{\geqslant 0}\,,\ \big\|\varphi(f,n,d) - u_d\big\|_2 \ \leqslant \ \big(\gamma(f,d)\big)^n \sqrt{\frac{d-1}{d}}\ . \tag{2}$$

The assertion of Gnacadja [1, Theorem 1], namely that $\lim_{n\to\infty} \varphi(f,n,d) = u_d$, is now a corollary of Theorem 1 above. The introduction in this earlier paper describes a general combinatorial problem this result is relevant to. We add here Figure 1 to visually convey the idea of this problem. The model studied in the main article is of this kind with $\ell = 2$.

We proceed to proving Theorem 1. We use classic notions of matrix algebra which may be found for instance in Horn and Johnson [2], and known facts about circulant matrices which may be found in Kra and Simanca [3] and references therein. We begin with two intermediate results.

We extend the definition of φ for convenience as follows.

$$\varphi(g,n,d,r) \ := \sum_{k\,\in\, r+d\mathbb{Z}} g^{*n}(k) \ = \ \sum_{q\in\mathbb{Z}} g^{*n}(r+d\,q)$$

for any finitely supported $\mathbb{Z}$-indexed vector $g = \big(g(k)\big)_{k\in\mathbb{Z}}$ and $n \in \mathbb{Z}_{\geqslant 0}$. We then define the d-vector $\varphi(g,n,d)$ by

$$\varphi(g,n,d) \ \ := \ \ \big(\varphi(g,n,d,0),\ldots,\varphi(g,n,d,d-1)\big)\ .$$

Let $\Phi(g,d)$ be the circulant matrix associated with the vector $\varphi(g,1,d)$. By definition, $\Phi(g,d)$ is a $d\times d$ matrix, its top row is the vector $\varphi(g,1,d)$, and each subsequent row is obtained from the preceding one by circularly shifting the entries rightward.

Lemma 2. *Let $g = \big(g(k)\big)_{k\in\mathbb{Z}}$ be finitely supported and let $d \in \mathbb{Z}_{\geqslant 1}$ and $n \in \mathbb{Z}_{\geqslant 0}$. Then, the matrix $\big(\Phi(g,d)\big)^n$ is the circulant matrix associated with the vector $\varphi(g,n,d)$. In particular, the top row of the matrix $\big(\Phi(g,d)\big)^n$ is the vector $\varphi(g,n,d)$, i.e.*

$$\varphi(g,n,d) \ = \ e_d \cdot \big(\Phi(g,d)\big)^n\ .$$

Proof. Let $\Psi(g,n,d) = \big(\Psi(g,n,d,r,s)\big)_{0\leqslant r,s\leqslant d-1}$ be the circulant matrix associated with the vector $\varphi(g,n,d)$. The claim in Lemma 2 is that $\big(\Phi(g,d)\big)^n = \Psi(g,n,d)$. The entries of $\Psi(g,n,d)$ are as follows.

$$\Psi(g,n,d,r,s) \ = \ \varphi\big(g,n,d,(s-r)\,\texttt{mod}\,d\big) \ = \sum_{k\,\in\, s-r+d\mathbb{Z}} g^{*n}(k)\ .$$

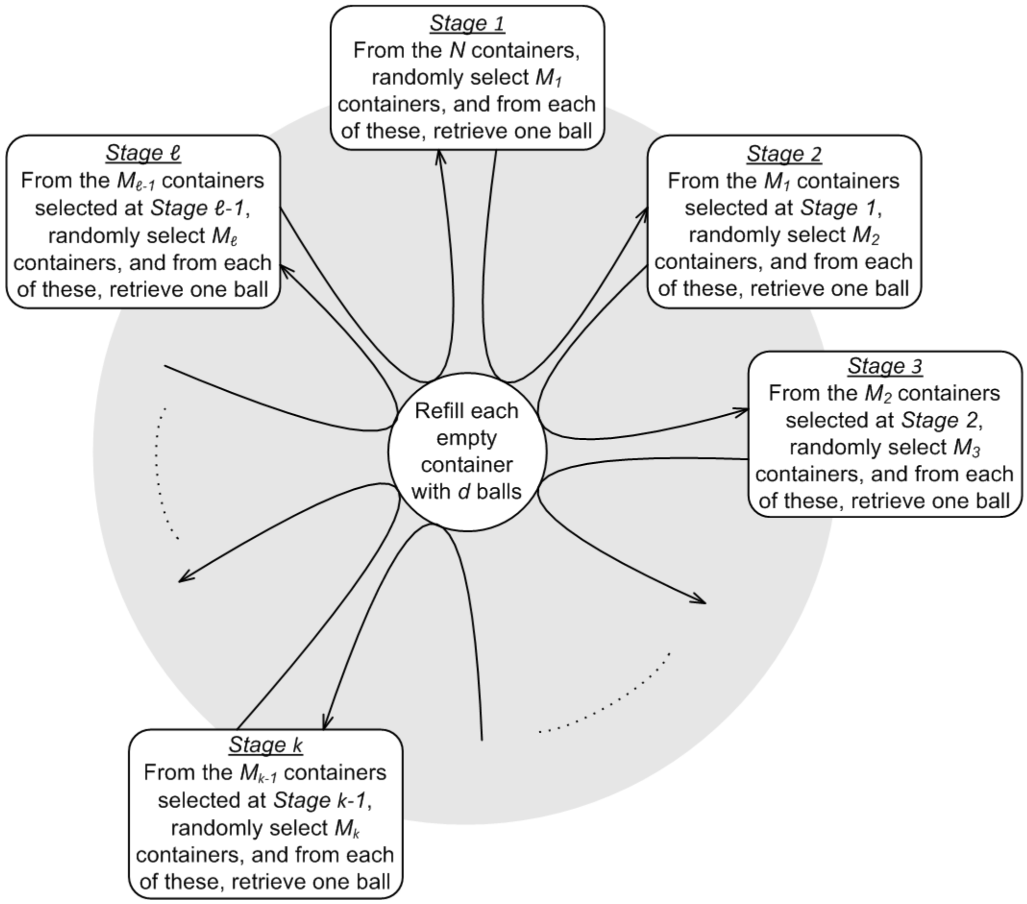

Figure 1: The ℓ-stage retrieval and replenishment process, which is the motivation and target application of this work. There are N containers and each is initially filled with d balls. With $M_0 := N$ and $M_{\ell+1} := 0$, the probability vector $f = \big(f(0), \ldots, f(\ell)\big)$ of Theorem 1 is given by $f(k) = (M_k - M_{k+1})/N$. We have $\ell = 2$ in the model studied in the main article.

Using this we have

$$\begin{aligned}
\Psi(g, m+n, d, r, t) &= \sum_{j \in t-r+d\mathbb{Z}} g^{*(m+n)}(j) \\
&= \sum_{j \in t-r+d\mathbb{Z}} \big(g^{*m} * g^{*n}\big)(j) \\
&= \sum_{j \in t-r+d\mathbb{Z}} \sum_{i \in \mathbb{Z}} g^{*m}(i) g^{*n}(j-i) \\
&= \sum_{i \in \mathbb{Z}} g^{*m}(i) \sum_{j \in t-r+d\mathbb{Z}} g^{*n}(j-i)
\end{aligned}$$

$$= \sum_{s=0}^{d-1} \sum_{i \in s-r+d\mathbb{Z}} g^{*m}(i) \sum_{j \in t-r+d\mathbb{Z}} g^{*n}(j-i)$$

$$= \sum_{s=0}^{d-1} \sum_{i \in s-r+d\mathbb{Z}} g^{*m}(i) \sum_{j \in t-r-i+d\mathbb{Z}} g^{*n}(j)$$

$$= \sum_{s=0}^{d-1} \sum_{i \in s-r+d\mathbb{Z}} g^{*m}(i) \sum_{j \in t-s+d\mathbb{Z}} g^{*n}(j)$$

$$= \sum_{s=0}^{d-1} \Psi(g,m,d,r,s)\Psi(g,n,d,s,t)\ .$$

Therefore,

$$\forall\, m,n \in \mathbb{Z}_{\geqslant 0}, \Psi(g,m+n,d) = \Psi(g,m,d) \cdot \Psi(g,n,d)\,.$$

This implies (and in fact is equivalent to)

$$\forall\, n \in \mathbb{Z}_{\geqslant 0}, \Psi(g,n,d) = \big(\Psi(g,1,d)\big)^n\,.$$

But $\Psi(g,1,d) = \Phi(g,d)$ because both these matrices are the circulant matrix associated with the vector $\varphi(g,1,d)$. Therefore,

$$\forall\, n \in \mathbb{Z}_{\geqslant 0}, \Psi(g,n,d) = \big(\Phi(g,d)\big)^n\,.$$

This completes the proof of Lemma 2. □

Lemma 3. *Let $g = \big(g(k)\big)_{k\in\mathbb{Z}}$ be finitely supported and nonnegative. Suppose that $g(0) > 0$ and $g(1) > 0$. Then the matrix $\Phi(g,d)$ is primitive.*

Proof. The matrix $\Phi(g,d)$ is nonnegative, so the assertion that it is primitive is equivalent to the existence of $n \in \mathbb{Z}_{\geqslant 1}$ such that the matrix $\big(\Phi(g,d)\big)^n$ is positive. Thanks to Lemma 2, this in turn is equivalent to the existence of $n \in \mathbb{Z}_{\geqslant 1}$ such that the vector $\varphi(g,n,d)$ is positive. Suppose that $g(k) > 0$ for $k = 0,\ldots,\ell$ for some $\ell \geqslant 1$. (We have $\ell = 1$ in Lemma 3, but the generality in the proof is intended to point out that one gets a smaller n with a larger ℓ.) Then $g^{*n}(k) > 0$ for $k = 0,\ldots,n\ell$. Suppose $n \geqslant (d-1)/\ell$. Let $r = 0,\ldots,d-1$. We have $0 \leqslant r \leqslant n\ell$, so $g^{*n}(r) > 0$. But $\varphi(g,n,d,r) \geqslant g^{*n}(r)$. So $\varphi(g,n,d,r) > 0$. Hence, the vector $\varphi(g,n,d)$ is positive. The proof of Lemma 3 is complete. □

Proof of Theorem 1.

Where a $\mathbb{Z}$-indexed vector is expected and we put f, one should read $\bar{f} = \big(\bar{f}(k)\big)_{k\in\mathbb{Z}}$, the vector extending f with zeros over $\mathbb{Z}$. Recall that $\Phi(f,d)$ is the circulant matrix associated with the vector $\varphi(f,1,d)$. As defined, $P_{f,d}$ is what is known as the representer polynomial of $\Phi(f,d)$. For $r = 0,\ldots,d-1$, let

$$\lambda_{f,d,r} \ :=\ P_{f,d}\left(\omega_d^{-r}\right) \qquad \text{and} \qquad v_{d,r} \ :=\ \frac{1}{\sqrt{d}}\left(1, \omega_d^r, \omega_d^{2r}, \ldots, \omega_d^{(d-1)r}\right)\ .$$

The following is well known: the eigenvalues of $\Phi(f,d)$ are $\lambda_{f,d,0}, \lambda_{f,d,1}, \ldots, \lambda_{f,d,d-1}$; $v_{d,r}$ is a left $\lambda_{f,d,r}$-eigenvector of $\Phi(f,d)$; and the vectors $v_{d,0}, v_{d,1}, \ldots, v_{d,d-1}$ form an orthonormal basis of $\mathbb{C}^n$. Note that $\lambda_{f,d,0} = 1$ and $v_{d,0} = \big(1/\sqrt{d}\big) u_d$.

The vector $\varphi(f,d)$ is a probability vector, so $\Phi(f,d)$ is nonnegative and all its row sums and column sums equal one; $\Phi(f,d)$ is doubly stochastic. It is also primitive by Lemma 3. By application of Perron-Frobenius theory, we obtain that the eigenvalue $\lambda_{f,d,0} = 1$ is simple and that for $r = 1, \ldots, d-1$, $|\lambda_{f,d,r}| < 1$. Therefore, $\gamma(f,d) < 1$.

Let

$$\mathcal{H}_d \;:=\; \left\{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : z_1 + \cdots + z_d = 0 \right\} .$$

This is a hyperplane in $\mathbb{C}^d$ containing the eigenvectors $v_{d,1}, \ldots, v_{d,d-1}$. Therefore, these $d-1$ vectors form an orthonormal basis of $\mathcal{H}_d$, $\mathcal{H}_d$ is stable under $\Phi(f,d)$, and

$$\forall\, z \in \mathcal{H}_d\,,\; \big\| z \cdot \Phi(f,d) \big\|_2 \;\leqslant\; \gamma(f,d)\, \|z\|_2 \,.$$

It follows that

$$\forall\, n \in \mathbb{Z}_{\geqslant 0}\,,\; \forall\, z \in \mathcal{H}_d\,,\; \Big\| z \cdot \big(\Phi(f,d)\big)^n \Big\|_2 \;\leqslant\; \big(\gamma(f,d)\big)^n\, \|z\|_2 \,.$$

Using Lemma 2, we obtain

$$\varphi(f,n,d) - u_d \;=\; e_d \cdot \big(\Phi(f,d)\big)^n - u_d \cdot \big(\Phi(f,d)\big)^n \;=\; \big(e_d - u_d\big) \cdot \big(\Phi(f,d)\big)^n \,.$$

We have $e_d - u_d \in \mathcal{H}_d$ and $\big\|e_d - u_d\big\|_2 = \sqrt{(d-1)/d}$, so

$$\forall\, n \in \mathbb{Z}_{\geqslant 0}\,,\; \big\|\varphi(f,n,d) - u_d\big\|_2 \;\leqslant\; \big(\gamma(f,d)\big)^n \big\|e_d - u_d\big\|_2 \;=\; \big(\gamma(f,d)\big)^n \sqrt{(d-1)/d}\,.$$

The proof of Theorem 1 is complete. □

References

[1] G. Gnacadja, *Asymptotic Equidistribution of Congruence Classes with respect to the Convolution Iterates of a Probability Vector*, Statistics and Probability Letters **82** (2012), no. 10, 1849–1852, DOI 10.1016/j.spl.2012.05.025.

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